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NORMAL FORM AND STABILITY OF PERIODIC SYSTEMS WITH INTERNAL RESONANCE

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We examine the problem of the Liapunov stability of the zero solution of a multidimensional nonlinear differential equation system with periodic coefficients and holomorphic right-hand sides. We investigate the critical case when the characteristic equation of the linearized system has only complex-conjugate roots equal to unity in modulus and when definite integral relations (internal resonance) exist between the characteristic indices and the frequency of the unperturbed motion. For any type of resonance we give a normal form of the system by reducing the original problem to the problem of stability under internal resonance for autonomous systems, considered earlier in [1]; this enables us to obtain necessary and sufficient conditions for the stability of a model system, i.e. for the most important cases of odd-order resonance. We show that in the majority of cases the instability of the model system implies the instability of the complete system. We show that an even-order resonance can lead to the asymptotic stability of the system. For a second-order system we give necessary and sufficient conditions for asymptotic stability with respect to the first nonlinear term in the normal form. We indicate the extension of the results obtained to the stability of Hamiltonian systems.

Let us consider the stability problem for the trivial solution of the system of equations

$$dx_* / dt = X_* (x_*, t) \tag{0.1}$$
$$X_* (0, t) \equiv 0, \quad X_* (x_*, t) = A(t)\, x_* + \sum_{l=m\geqslant 2}^{\infty} X_*^{(l)} (x_*, t)$$

Here x_* and $X_* \in E_{2n}$ and $X_* (x_*, t)$ is an analytic vector function periodic in time t with a real period ω. We investigate the critical case when the matrix $A(t)$ is such that all the roots of the characteristic equation are complex and equal to unity in modulus. Then, according to [2], the stability problem for the trivial solution of the nonautonomous system (0.1) reduces to the critical case of the stability of n pairs of purely imaginary roots for an autonomous system if no relations of the form

$$\langle \Lambda P \rangle = \frac{2\pi i}{\omega} p, \quad i = \sqrt{-1}, \quad p = 0, \pm 1 \pm 2, \ldots \tag{0.2}$$
$$\Lambda = (\lambda_1, \ldots, \lambda_n), \; P = (p_1, \ldots, p_n)$$
$$|P| = p_1 + \ldots + p_n = k \geqslant 3, \; p_s \geqslant 0$$

exist detween the characteristic indices $\pm \lambda_s$ ($\lambda_s^2 < 0$, $s = 1, 2, \ldots, n$) and the number $2\pi i / \omega$. Here Λ is the vector of the system's characteristic indices, P is an n-dimensional vector with integral components, $p_1, \ldots, p_n$ are relatively prime integers, including zero.

Definition. We say that the system (0.1) possesses a kth-order internal resonance if relations (0.2) are satisfied.

Our aim is to investigate the stability of the trivial solution of system (0.1) with internal resonance. In all the subsequent discussions we are restricted by the following assumptions: (1) all λ_s are distinct (certain cases of multiple frequencies were considered in [3]); (2) none of the ratios $\lambda_s \omega i/\pi$, $s = 1, 2, \ldots, n$ is an integer (otherwise, the characteristic equation would have roots equal to ± 1); (3) for a specified Λ there exists a unique resonance vector P satisfying conditions (0.2), i.e. there is no interaction of resonances when the same frequencies occur simultaneously in several resonance relations.

According to [2, 4] we can write system (0.1) in the form

$$x^\cdot = \lambda x + \sum_{l=m\geqslant 2}^{\infty} X^{(l)}(x, y, t), \quad y^\cdot = -\lambda y + \sum_{l=m\geqslant 2}^{\infty} Y^{(l)}(x, y, t) \tag{0.3}$$

$$x = (x_1, \ldots, x_n), \quad y = (y_1, \ldots, y_n), \quad \lambda = \{\lambda_1, \ldots, \lambda_n\}$$

by using a nonsingular linear transformation with periodic coefficients. Here x and y are complex-conjugate vectors, λ is a diagonal matrix, $X^{(l)}(x, y, t)$ and $Y^{(l)}(x, y, t)$ are complex-conjugate vector functions periodic in t with period ω, whose components $X_s^{(l)}$ and $Y_s^{(l)}$ are represented by l th-order forms, so that

$$X_s^{(l)} = \sum_{|k_s|+|l_s|=l} R_{k_s l_s}(t) x_1^{k_{s1}} \ldots x_n^{k_{sn}} y_1^{l_{s1}} \ldots y_n^{l_{sn}}$$

$$R_{k_s l_s}(t) = R_{k_s l_s}(t+\omega), \quad k_s = (k_{s1}, \ldots, k_{sn}) \quad l_s = (l_{s1}, \ldots, l_{sn})$$

$$|k_s| = k_{s1} + \ldots + k_{sn}, \quad |l_s| = l_{s1} + \ldots + l_{sn}, \quad k_{sj}, l_{sj} \geqslant 0;$$
$$j, s = 1, 2, \ldots, n$$

where k_s and l_s are all possible integral vectors. The normal form [5] of system (0.3) is of great value in what follows.

1. Reduction to normal form. Let us show that by using a nonlinear (polynomial) Liapunov transformation with periodic coefficients we can reduce system (0.3) to a form in which the nonlinear terms of arbitrarily high order do not contain time t explicitly. To do this we transform system (0.3) by the substitution

$$x = u + \sum_{l=m}^{2N+1} U^{(l)}(u, v, t), \quad y = v + \sum_{l=m}^{2N+1} V^{(l)}(u, v, t)$$

where N is an arbitrarily large positive integer and $U^{(l)}$ and $V^{(l)}$ are complex-conjugate vector functions whose components $U_s^{(l)}$ and $V_s^{(l)}$, $s = 1, 2, \ldots, n$ are lth-order forms with coefficients periodic in t of period ω, so that

$$U_s = \sum_{|k_s|+|l_s|=l} U_{k_s l_s}(t) u_1^{k_{s1}} \ldots u_n^{k_{sn}} v_1^{l_{s1}} \ldots v_n^{l_{sn}}$$

According to [3, 4], the functions $U_{k_s l_s}$ are found from the equations

$$\frac{dU_{k_s l_s}}{dt} + \varkappa_{k_s l_s} U_{k_s l_s} = \Phi_{k_s l_s} - C_{k_s l_s} \tag{1.1}$$

$$\varkappa_{k_s l_s} = \langle (k_s - l_s)\Lambda \rangle - \lambda_s, \quad |k_s| + |l_s| = l$$

Here $C_{k_s l_s}$ are the coefficients in the l th-order forms of the transformed equations for u_s and v_s, and $\Phi_{k_s l_s}$ are known functions of $U_{k_s l_s}$ and $V_{k_s l_s}$ with $|k_s| + |l_s| \leqslant l - 1$.

When $l = m$ the functions $\Phi_{k_s l_s}$ turn into $R_{k_s l_s}$, i.e. are specified periodic functions of t with period ω. We omit the subscripts k_s and l_s in what follows.

Two cases can occur for Eq.(1.1). In the first one let the condition

$$\varkappa = \frac{2\pi i}{\omega} q, \quad q = 0, \pm 1, \pm 2, \ldots \tag{1.2}$$

not be satisfied. The terms of the transformed equations, corresponding to such vectors k_s and l_s, are called nonresonance terms. Then, according to [4], for any a priori specified constants C there exists a unique periodic solution (of period ω) of Eq.(1.1)

$$U = e^{-\varkappa t}\left[\frac{e^{-\varkappa\omega}}{1 - e^{-\varkappa\omega}}\int_0^\omega e^{\varkappa t}\Phi(t)\,dt + \int_0^t e^{\varkappa t}\Phi(t)\,dt\right] \tag{1.3}$$

We can thus suppress all nonresonance terms by setting $C = 0$.

Let us now consider the vectors k_s and l_s for which condition (1.2) is satisfied. The terms of the transformed equations, corresponding to these vectors, are called resonance terms. In this case the solution (1.3) for U loses its meaning. However if we set

$$C = e^{-\varkappa t}c, \qquad c = \frac{1}{\omega}\int_0^\omega \Phi(t)\,e^{\varkappa t}dt \tag{1.4}$$

then the functions $U(t)$ will, as before, be periodic for all resonance vectors k_s and l_s In fact, using the Fourier expansion for $\Phi(t)$

$$\Phi(t) = \sum_{\nu=-\infty}^{+\infty} b_\nu \exp\left(\frac{2\pi i}{\omega}\nu t\right)$$

we can present the general solution of Eq.(1.1) as

$$U(t) = e^{-\varkappa t}\left\{B + \int_0^t\left[\sum_{\nu=-\infty}^{+\infty} b_\nu \exp\left(\frac{2\pi i}{\omega}\nu + \varkappa\right)t - c\right]dt\right\}$$

Here c is chosen according to (1.4), while B is an arbitrary constant. Then, for $\varkappa = 2q\pi i/\omega$ we obtain

$$U(t) = e^{-\varkappa t}\left\{B + \sum_{\nu=-\infty}^{+\infty} b_\nu \frac{\omega\exp[2\pi i\omega^{-1}(\nu + q)\,t]}{2\pi i\,(\nu + q)}\right\}, \quad \nu \neq -q$$

i.e. $U(t)$ is an ω-periodic function of t for any value of B. But if all U become ω-periodic for $|k_s| + |l_s| = m$, then for $|k_s| + |l_s| = m + 1$ all $\Phi(t)$ occurring in Eq.(1.1) will be specified ω-periodic functions of t and, consequently, for the determination of the functions U at the next step of the transformation we obtain equations of the same form as those for $|k_s| + |l_s| = m$. Thus, for arbitrarily large positive integer $l = |k_s| + |l_s|$, all functions $U(t)$ satisfying Eqs.(1.1) will be ω-periodic for $|k_s| + |l_s| \leqslant l$.

Let us elucidate the structure of the transformed equations. Obviously, the C are constant for those k_s and l_s which make $\varkappa$ zero. It is easy to see that the relation $\varkappa = 0$ is fulfilled identically with respect to Λ for every odd l if

$$k_{sj} = l_{sj} + \delta_{sj}, \qquad s, j = 1, 2, \ldots, n$$

where δ_{sj} is the Kronecker symbol. The terms corresponding to these resonance vectors k_s and l_s are called identity responance terms. We note that $\varkappa$ can vanish independently of the parity of l if Λ satisfies the internal resonance condition (0.2) with $p=0$.

The coefficients C for the corresponding resonance vectors k_s and l_s also become constants and are determined by formula (1. 4). However, it is advisable to compute such resonance vectors by examining the general case of (1. 2) which includes the internal resonance conditions (0. 2).

For an l-th order form relations (1. 2) obviously coincide with (0. 2) in the two cases

$$1)\ l_{sj} = \varepsilon p_j + h_{sj} - \delta_{sj},\quad k_{sj} = h_{sj},\quad q_{k_s l_s} = -\varepsilon p$$

$$2)\ k_{sj} = \varepsilon p_j + h_{sj} + \delta_{sj},\quad l_{sj} = h_{sj},\ q_{k_s l_s} = \varepsilon p$$

Here h_{sj} are all possible nonnegative integers such that

$$\varepsilon k + 2\sum_{j=1}^{n} h_{sj} = l \pm 1$$

(the plus sign is taken in the first case, the minus, in the second case, while ε takes all positive integer values, $\varepsilon = 1, 2, \ldots, \varepsilon'$, where ε' is the largest integer contained in the fraction $(l + 1) / k$ in the first case and in $(l - 1)/k$ in the second case.

Hence we can deduce, and this is important in what follows, that kth-order internal resonance terms can appear only in the forms of not lower than ($k - 1$)-st order. Thus, in the variables u_s and v_s the first group of complex-conjugate equations take, to within terms of ($2N + 1$)-st order the form

$$v_s u_s^{\cdot} = \lambda_s r_s + r_s \sum_{l\geqslant m}^{2N+1} \sum_{2|k_s|=l-1} c_{k_{sj}} \prod_{j=1}^{n} r_j^{k_{sj}} + \tag{1.5}$$
$$\sum_{l=k-1}^{2N+1} \sum_{\varepsilon=1}^{\varepsilon_1} \exp\left(\frac{2\pi i}{\omega}\varepsilon pt\right) \prod_{j=1}^{n} v_j^{\varepsilon p_j} \sum_{2|h_s|=l+1-\varepsilon k} c_{h_{sj}}^{(\varepsilon)} \prod_{j=1}^{n} r_j^{h_{sj}} +$$
$$r_s \sum_{l=k+1}^{2N+1} \sum_{\varepsilon=1}^{\varepsilon_2} \exp\left(-\frac{2\pi i}{\omega}\varepsilon pt\right) \prod_{j=1}^{n} u_j^{\varepsilon p_j} \sum_{2|h_s|=l-1-\varepsilon k} c_{h_{sj}}^{(\varepsilon)} \prod_{j=1}^{n} r_j^{h_{sj}} + \ldots$$
$$c_{h_{sj}}^{(\varepsilon)} = \frac{1}{\omega}\int_0^{\omega} \Phi_{h_{sj}}(t) \exp\left(\pm\frac{2\pi i}{\omega}\varepsilon pt\right) dt,\qquad r_s = u_s v_s,\qquad s = 1, 2, \ldots, n$$

Here ε_1 and ε_2 are the largest integers contained in the fractions $(l + 1)/k$ and $(l - 1)/k$, respectively; the minus sign is taken for the first group of internal resonance terms and the plus, for the second group ; terms of not less than order $2(N + 1)$ relative to $r_1, \ldots, r_n$ are not written out.

In order to eliminate time in the internal resonance terms of system (1. 5), we pass to the new variables ξ_s and η_s from the variables u_s and v_s , and we obtain

$$\eta_s \xi_s^{\cdot} = r_s \sum_{l\geqslant m}^{2N+1} \sum_{2|k_s|=l-1} c_{k_{sj}} \prod_{j=1}^{n} r_j^{k_{sj}} + \tag{1.6}$$
$$\sum_{l=k-1}^{2N+1} \sum_{\varepsilon=1}^{\varepsilon_1} \prod_{j=1}^{n} \eta_j^{\varepsilon p_j} \sum_{2|h_s|=l+1-\varepsilon k} c_{h_{sj}}^{(\varepsilon)} \prod_{j=1}^{n} r_j^{h_{sj}} +$$
$$r_s \sum_{l=k+1}^{2N+1} \sum_{\varepsilon=1}^{\varepsilon_2} \prod_{j=1}^{n} \xi_j^{\varepsilon p_j} \sum_{2|h_s|=l-1+\varepsilon k} c_{h_{sj}}^{(\varepsilon)} \prod_{j=1}^{n} r_{j_1}^{h_{sj}} + \ldots$$

$$u_s = \xi_s e^{\lambda_s t}, \quad v_s = \eta_s e^{-\lambda_s t}, \quad s = 1, 2, \ldots, n$$

By separating the real and imaginary parts, system (1.6) is reduced to a $2n$th-order system in the critical case of $2n$ zero roots with $2n$ groups of solutions (but not with n zero roots as in the nonresonance case) with constant coefficients, up to terms of $(2N+1)$-st order, inclusive. The system obtained from (1.6) by discarding terms of higher than $(2N+1)$-st order is called the model system.

Note. The normal form obtained for the model system shows that it coincides completely with the normal form for autonomous systems [1] and remains the same whether or not resonance obtains only between the system's characteristic indices $\lambda_1, \ldots, \lambda_n$ or between these indices and the frequency of the unperturbed periodic motion.

On the basis of the above reasoning we can state the following theorem.

Theorem 1. The stability problem for the periodic motion of systems of higher than second order with internal resonance, when it is solved by a finite number of terms of the equations of perturbed motion, reduces completely to the equilibrium stability problem for autonomous systems with internal resonance of the same order.

Note. The case, most important in practice, of odd-order resonance in a second-order system was considered in [6]. Even-order resonance is considered below.

The normal form obtained enables us to distinguish two of the simplest and at the same time most important cases of the solution to the stability problem. One of them corresponds to an odd-order resonance with $k = m + 1$ for a system of arbitrary order $(n \geqslant 1)$, while the other corresponds to an even-order resonance with $n = 1$. We pass on to a more detailed consideration of these cases.

2. Odd-order resonance. Identity resonance terms are absent in system (1.6) when k is odd. However, in the general case of $m > k - 1 \geqslant 2$ the structure of the resonance terms still remains very complex for a complete solution of the problem. The case $m = k - 1 \geqslant 2$ is most important in practice. A complete solution of this problem for a model system was given in [1] wherein necessary and sufficient conditions were obtained, from which it follows that the model system either can preserve the neutrality of the linear approximation or can become unstable. Therein it was shown that the instability of the model system, except for certain degenerate cases, necessarily implies the instability of the complete system. Obviously, the latter result is valid also for the periodic motion stability problem being investigated since in comparison with the autonomous systems considered in [1] the terms of higher than $(2N+1)$-st order, contained in Eqs. (1.6), depend periodically on t and, consequently, are bounded functions in a sufficiently small neighborhood of the origin, as required in the proof of instability given in [1]. From what has been stated it follows also that the results obtained can be partially extended to the stability of the periodic motions of the Hamiltonian systems.

3. Even-order resonance. In this case the model system becomes more complex because of the presence of identity resonance terms in it. This does not permit us to obtain a complete solution for systems of higher than second order. However, for second-order systems the original problem reduces to the critical case of two zero roots with two groups of solutions, considered in detail in [3].

It is interesting, however, to ascertain which of the possible versions of the solutions in the critical case (stability, instability, asymptotic stability) are realized under internal resonance; it is also interesting to obtain the stability conditions directly from the coefficients of the normal form.

Let us solve the problem for the most important case of fourth-order resonance. When $n = 1$ the internal resonance conditions acquire the form $2\lambda\omega = p\pi i$; on the basis of the assumptions made above, p can take odd values only. The normal form of the system is

$$\xi^{\cdot} = c_1\xi^2\eta + c_2\eta^3 + O(\xi\eta)^2$$

(we have not written down the complex-conjugate equation for η). Here c_1, and c_2 are complex coefficients determinable from formulas (1.4) in terms of the coefficients of the original system. By introducing the polar coordinates r and θ, we arrive at a system of the form

$$r^{\cdot} = r^3\sqrt{a_2^2 + b_2^2}\,[A + \cos(\psi - 4\theta)] + \ldots \qquad (3.1)$$

$$r\theta^{\cdot} = r^3\sqrt{a_2^2 + b_2^2}\,[B + \sin(\psi - 4\theta)] + \ldots$$

Here

$$A = \frac{a_1}{\sqrt{a_2^2 + b_2^2}}, \quad B = \frac{b_1}{\sqrt{a_2^2 + b_2^2}}, \quad \sin\psi = \frac{b_2}{\sqrt{a_2^2 + b_2^2}},$$

$$\cos\psi = \frac{a_2}{\sqrt{a_2^2 + b_2^2}}$$

$$\xi = re^{i\theta}, \quad \eta = re^{-i\theta}, \quad c_s = a_s + ib_s, \quad s = 1, 2$$

terms of not lower then fourth order in r are not written out.

The stability problem for system (3.1) is completely solved by Kamenkov's theorem [3] according to which the trivial solution of system (3.1) is unstable if the form

$$R(\theta_*) \equiv r^4\sqrt{a_2^2 + b_2^2}\,[A + \cos(\psi - 4\theta_*)] > 0$$

for even one value of θ_*, being a root of the equation

$$\Phi(\theta) \equiv r^4\sqrt{a_2^2 + b_2^2}\,[B + \sin(\psi - 4\theta)] = 0 \qquad (3.2)$$

If, however, $R(\theta_*) < 0$ for all solutions of Eq. (3.2), then the trivial solution is asymptotically stable. For a sign-definite $\Phi(\theta)$ the stabilitv question is resolved by the sign of the integral

$$G = \frac{1}{2\pi}\int_0^{2\pi} \frac{R(\theta)}{\Phi(\theta)}\,d\theta$$

namely: instability if $G > 0$ and asymptotic stability if $G < 0$. When $R(\theta_*) \leqslant 0$ or $G \equiv 0$ the stability problem is resolved by the terms of higher than third order in r.

Obviously, when $|B| < 1$ all values of θ making $R(\theta)$ zero are found from the equation $\sin(4\theta - \psi) = B$. On these solutions the form $R(\theta)$ takes the values

$$R = r^4\sqrt{a_2^2 + b_2^2}\,(A \pm \sqrt{1 - B^2})$$

and can be both positive as well as negative. When $|B| > 1$ the form $\Phi(\theta)$ becomes sign-definite. By elementary calculations we convince ourselves that $G \equiv 0$ in this case, and, consequently, the stability problem cannot be resolved by the third-order terms of system (3.1). The situation is similar when $A = -\sqrt{1 - B^2}$ since form R vanishes on one of the solutions of Eq. (3.2) and is negative on the other one.

On the basis of the above reasoning we can state the following theorem giving the necessary and sufficient conditions for the asymptotic stability with respect to the first nonlinear terms in system (0.1) for $n = 1$ with a fourth-order resonance.

Theorem 2. For the asymptotic stability of the trivial solution of system (3.1) (and, consequently, of the original system (0.1) with $n = 1$ and fourth-order resonance), it is necessary and sufficient to satisfy the inequalities

$$a_1 < -\sqrt{a_2^2 + b_2^2 - b_1^2}$$
$$(a_1 \neq -\sqrt{a_2^2 + b_2^2 - b_1^2}, \quad b_1^2 < a_2^2 + b_2^2)$$

Note. The instability conditions given by the theorem extend, in particular, to Hamiltonian systems. In fact, as is easily verified, from the canonicity conditions for system (3.1) it follows that $a_1 = 0$, therefore, on the basis of the theorem proved, we conclude that the periodic solution of the original canonic system is unstable when $b_1^2 < a_2^2 + b_2^2$.

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STABILIZATION IN A LINEAR DIFFERENTIAL GAME WITH INTEGRAL CONSTRAINTS ON THE PLAYERS' CONTROL RESOURCES

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We examine an encounter game problem for a linear controlled system. We assume that the controls of the first and second players are subject to integral constraints. Using the idea of control with a guide [1] and the methods of stabilization theory [2], we construct a stabilized control procedure ensuring a stable encounter of the motions generated by it with a specified target set. The contents of the paper is related to the investigations in [1, 3 — 5].

1. Let the motion of a controlled system be described by the vector differential equation

$$dx / dt = Ax + Bu + Cv, \quad x[t_0] = x_0 \tag{1.1}$$